

M-branes at angles

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Supersymmetric configurations of non-orthogonally intersecting M-5-branes can be obtained by rotation of one of a pair of parallel M-5-branes. Examples preserving 1/4, 3/16 and 1/8 supersymmetry are reviewed.

1. INTRODUCTION

In contrast to the current rapid evolution in our understanding of the microscopic degrees of freedom of M-theory, the macroscopic picture has changed very little over the past year. The ‘basic’ ingredients remain D=11 supergravity and its 1/2-supersymmetric solutions: the M-wave, the M-2-brane, the M-5-brane and, in the S^1 Kaluza-Klein vacuum, the M-monopole (i.e. Euclidean-Taub-Nut times 7-dimensional Minkowski). The novelty has been in the way that these ingredients have been combined to form new ‘intersecting-brane’ configurations, preserving some smaller fraction of supersymmetry, and in the remarkable insights into quantum field theory that these configurations have led to. I shall have nothing to say here about these applications of intersecting brane configurations. Instead, I shall discuss one aspect of attempts towards a classification of them: the conditions for partial preservation of supersymmetry by configurations of *non-orthogonally* intersecting M-branes.

While it is not difficult to see why orthogonal intersections of branes can partially preserve supersymmetry, it is less obvious that this is also true for non-orthogonal intersections. This possibility was first pointed out by Berkooz, Douglas and Leigh [1], who also provided several D-brane examples, which they interpreted as rotations within an $SU(n)$ subgroup of $SO(2n)$. Further examples, interpretable as rotations within an $Sp(2)$ subgroup of $SO(8)$, were given by Gauntlett, Gibbons, Papadopoulos and the author [2] and shown to be related via M-theory dualities to 8-dimensional hyper-Kähler manifolds. I shall re-

view both cases here although the interpretation as rotations within subgroups of reduced holonomy and the connection with (hyper)complex geometry will not be explained. Instead I shall concentrate on clarifying some aspects of the Dirac matrix algebra needed to determine the fraction of supersymmetry preserved by rotated M-brane configurations, i.e. ‘M-branes at angles’.

It is convenient to start with two parallel M-5-branes. Essentially no generality is lost by considering only M-5-branes, and any non-parallel configuration of two intersecting M-5-branes can be obtained by some rotation of one of them from the parallel configuration. We may assume that the two parallel M-5-branes lie in the 12349 5-plane. This configuration can be represented by the array

$$\begin{array}{cccccccccc} M : & 1 & 2 & 3 & 4 & - & - & - & - & 9 & - \\ M : & 1 & 2 & 3 & 4 & - & - & - & - & 9 & - \end{array}$$

and is associated with the constraint

$$\Gamma_{091234}\epsilon = \epsilon \tag{1}$$

where ϵ is the asymptotic value of a Killing spinor of the corresponding supergravity solution. Here we shall refer to the solutions of this and similar algebraic equations as ‘Killing spinors’. Since Γ_{091234} is traceless and squares to the identity, the space of solutions of (1) is 16-dimensional, i.e. the configuration preserves 1/2 supersymmetry. We now wish to rotate the second M-5-brane away from the 12349 5-plane to give a configuration of two intersecting M-5-branes. If the spinor representation of the rotation matrix is R then the constraint imposed by the presence of the second

M-5-brane is [1]

$$R\Gamma_{091234}R^{-1}\epsilon = \epsilon. \quad (2)$$

Although R may be a general $SO(10)$ matrix, the relative orientation of the second M-5-brane is actually determined by only five angles. To see this we observe [2] that the orientation of each M-5-brane is determined by a set of five linearly independent normals, so that the relative orientation is determined by the 5×5 matrix M of inner products of one set of normals with the other. Each set can be taken to be orthonormal. The remaining freedom in the choice of the two sets of normals allows the diagonalization of M by the action of $O(5; \mathbb{R}) \times O(5; \mathbb{R})$. The diagonal entries are the cosines of the five angles. In the spinor representation, the rotation matrix can be chosen to be¹

$$R = e^{\frac{1}{2}[\vartheta\Gamma_{15} + \psi\Gamma_{26} + \varphi\Gamma_{37} + \rho\Gamma_{48} + \zeta\Gamma_{9\mathfrak{k}}]} \quad (3)$$

where $\vartheta, \psi, \varphi, \rho$ and ζ are the five angles. Note that $\Gamma_{091234}R^{-1} = R\Gamma_{091234}$, so that (2) becomes $R^2\Gamma_{091234}\epsilon = \epsilon$. In view of (1), this means that the constraint imposed on Killing spinors by the presence of the second M-5-brane is equivalent to

$$[R^2 - 1]\epsilon = 0 \quad (4)$$

with R given by (3).

What we now have to do now is to determine for a given configuration, specified by the five angles, the number of solutions to the simultaneous equations (1) and (4), and hence the fraction ν of supersymmetry preserved by the configuration. Ultimately, we would like to determine ν as a (discontinuous) function of $\vartheta, \psi, \varphi, \rho, \zeta$, but this will not be attempted here. Instead, some particular cases preserving 1/4, 3/16 and 1/8 supersymmetry will be reviewed.

¹We use a base 11 arithmetic in which \mathfrak{k} is the symbol for the number 10. Since the standard notation for integers is based on the number of angles (none for 0, one for 1, etc) a symbol with ten angles would be ideal although impractical. The symbol ‘ \mathfrak{k} ’, to be pronounced ‘ten’, seems to be a reasonable compromise. In any case it is certainly a ‘natural’ choice.

2. 1/4 SUPERSYMMETRY

We begin the analysis by considering the case of a rotation by an angle ϑ in the 15 plane, for which

$$R^2 - 1 = [e^{\vartheta\Gamma_{15}} - 1]. \quad (5)$$

This has a zero eigenvalue only for $\vartheta = 0 \pmod{2\pi}$. Thus, a rotation in the 15-plane breaks supersymmetry. We therefore move on to consider a simultaneous rotation in the 15 and 26 planes. In this case one easily sees that $(R^2 - 1)$ can have zero eigenvalues only if $\psi = \pm\vartheta$. The sign is irrelevant, so set $\psi = \vartheta$ (and $\varphi = \rho = \zeta = 0$) in (3) to get the constraint

$$\begin{aligned} 0 &= [e^{\vartheta[\Gamma_{15} + \Gamma_{26}]} - 1]\epsilon \\ &= \sin\theta\Gamma_{15}e^{\vartheta\Gamma_{15}}(1 - \Gamma_{1526})\epsilon. \end{aligned} \quad (6)$$

This is an identity if $\vartheta = 0$, as expected, but it is also an identity if $\vartheta = \pi$. The reason for this is that while a rotation by π in the 15-plane just converts the M-5-brane into an anti-brane, the simultaneous rotation by π in the 26-plane returns it to the original M-5-brane. If $\sin\vartheta$ is non-zero then (6) is satisfied only if

$$\begin{aligned} \epsilon &= \Gamma_{1526}\epsilon \equiv \Gamma_{034569}\Gamma_{012349}\epsilon \\ &= \Gamma_{034569}\epsilon \end{aligned} \quad (7)$$

where the last line follows from (1). But this is just the constraint associated with an M-5-brane in the 34569 5-plane. Thus, as shown originally in the context of D-branes [1], whatever the value of the angle θ (other than zero mod 2π) the rotated configuration preserves the same fraction of supersymmetry as the orthogonal intersection of two M-5-branes on a 3-plane, represented by the array

$$\begin{array}{cccccccccc} M : & 1 & 2 & 3 & 4 & - & - & - & - & 9 & - \\ M : & - & - & 3 & 4 & 5 & 6 & - & - & 9 & - \end{array}$$

This fraction is 1/4 [3,4]. In fact, the orthogonal intersection of any pair of branes either breaks all supersymmetry or preserves 1/4 supersymmetry. One way to understand why certain rotations away from orthogonality can preserve the 1/4 supersymmetry of orthogonal intersections is

that the non-orthogonal rotations are related to the orthogonal ones by duality [5]. This explanation is, of course, not available for rotations which preserve less than 1/4 supersymmetry, although orthogonal intersections of more than two branes can yield 1/8, 1/16, and even [6] 1/32 supersymmetry.

The next case to consider in a systematic analysis of rotations of two M-5-branes would be simultaneous rotations in three independent planes. It is straightforward to see that partial preservation of supersymmetry in this case requires the three angles to sum to zero (mod 2π and for an appropriate choice of the signs). An example of this type was given in [1] in which two IIA 6-branes intersect on a 3-plane: the intersection preserves 1/8 supersymmetry if the relative orientation is obtained by a rotation within an $SU(3)$ subgroup of $SO(6)$. The M-5-brane dual of this configuration is two M-5-branes intersecting non-orthogonally on a 2-plane. The orthogonal intersection of this type would break all supersymmetry, but it cannot be reached by a rotation in which the three angles sum to zero. We shall skip the detailed analysis of this case and move on to simultaneous rotations in four independent planes. A complete analysis of the latter case will not be attempted either. Instead, some examples of special interest will be described, starting with one that preserves 3/16 supersymmetry, a fraction that cannot be realized by orthogonal intersections.

3. 3/16 SUPERSYMMETRY

By means of an explicit calculation in a particular representation of the Dirac matrices, the rotation matrix

$$R = e^{\frac{1}{2}\vartheta[\Gamma_{15} + \Gamma_{26} + \Gamma_{37} + \Gamma_{48}]} \quad (8)$$

was shown in [2] to lead to preservation of 3/16 supersymmetry. Here, I shall give an alternative, representation-independent, proof of the 3/16 supersymmetry. We first note that

$$R^2 = \prod_{i=1}^4 (\cos \vartheta + \sin \vartheta \Gamma_{i\,i+4}). \quad (9)$$

Carrying out the multiplications, we find that

$$R^2 - 1 = \sin^4 \vartheta (\Gamma_{12345678} - 1) + \frac{1}{8} \Sigma^2 \sin^2 2\vartheta + \frac{1}{2} \sin 2\vartheta (\cos^2 \vartheta - \Gamma_{12345678} \sin^2 \vartheta) \quad (10)$$

where

$$\Sigma = [\Gamma_{15} + \Gamma_{26} + \Gamma_{37} + \Gamma_{48}]. \quad (11)$$

As expected, $R^2 = 1$ when $\vartheta = 0, \pi$. When $\vartheta = \pi/2$ we have $R^2 = \Gamma_{12345678}$, so that ϵ must satisfy $\Gamma_{12345678}\epsilon = \epsilon$. In view of (1) this is equivalent to the constraint

$$\Gamma_{056789}\epsilon = \epsilon. \quad (12)$$

This corresponds to the orthogonal intersection of two M-5-branes on a line, summarized by the array

$$\begin{array}{cccccccccc} M : & 1 & 2 & 3 & 4 & - & - & - & - & 9 & - \\ M : & - & - & - & - & 5 & 6 & 7 & 8 & 9 & - \end{array}$$

This is as expected because a $\vartheta = \pi/2$ rotation yields precisely this configuration, which is known to preserve 1/4 supersymmetry [7].

For other values of ϑ we proceed as follows. The matrix Σ satisfies

$$\Sigma^2 + 4 = 2\Lambda \quad (13)$$

where

$$\begin{aligned} \Lambda = & \Gamma_{1526} + \Gamma_{1537} + \Gamma_{1548} \\ & + \Gamma_{2637} + \Gamma_{2648} + \Gamma_{3748}. \end{aligned} \quad (14)$$

The matrix Λ satisfies

$$\Lambda^2 + 4\Lambda - 6 = 6\Gamma_{12345678}. \quad (15)$$

It follows that Λ and $\Gamma_{12345678}$ are simultaneously diagonalizable. Moreover

$$\Gamma_{12345678}\Lambda \equiv \Lambda, \quad (16)$$

so all eigenvalues of Λ in the $(-)$ eigenspace of $\Gamma_{12345678}$ vanish. In the $(+)$ eigenspace (15) yields eigenvalues -6 and 2 . Since Λ is traceless the relative multiplicity of these non-zero eigenvalues is $1 : 3$.

It is convenient to now introduce the constraint (1). As Γ_{091234} commutes with Λ (and

with $\Gamma_{12345678}$) we can write $\Lambda = \Lambda_+ + \Lambda_-$, where Λ_{\pm} are the projections of Λ onto the (\pm) eigenspaces of Γ_{091234} . Now, both Λ and $\Gamma_{091234}\Lambda$ are traceless, so Λ_+ and Λ_- are traceless too. The relative multiplicities of the eigenvalues of Λ_+ are therefore the same as those of Λ . In other words, the eigenspinors of Λ in the 16-dimensional eigenspace of Γ_{091234} with eigenvalue $+1$ are again $0, -6, 2$ and the multiplicities are $8, 2, 6$. This translates to the following eigenvalues of Σ^2 :

$$-4(8, -), \quad -16(2, +), \quad 0(6, +). \quad (17)$$

The numbers in parentheses are the multiplicities of eigenvalues for eigenspinors satisfying (1) and the sign is the sign of the eigenvalue of $\Gamma_{12345678}$. Now (10) can be rewritten as

$$R_+^2 - 1 = \frac{1}{2}\Sigma \sin 2\vartheta (\cos 2\vartheta + \frac{1}{4}\Sigma \sin 2\vartheta) \quad (18)$$

$$R_-^2 - 1 = -2 \sin \vartheta \left[\sin \vartheta - \frac{1}{2}\Sigma \cos \vartheta - \frac{1}{8}(\Sigma^2 + 4) \cos \vartheta \sin 2\vartheta \right] \quad (19)$$

where the (\pm) subscript indicates the eigenspace of $\Gamma_{12345678}$. The zero eigenvalues of Σ^2 clearly correspond to zero eigenvalues of $R^2 - 1$ and it is straightforward to check that the non-zero eigenvalues of Σ^2 do not. There are therefore precisely 6 independent simultaneous solutions of (1) and (4), implying preservation of $3/16$ supersymmetry.

4. 1/8 SUPERSYMMETRY

We shall conclude this analysis by considering a case that, in general, preserves only $1/8$ supersymmetry but which includes both the previous $\nu = 3/16$ and $\nu = 1/4$ cases as special limits. We start from the rotation matrix

$$R = e^{\frac{1}{2}\vartheta(\Gamma_{15} + \Gamma_{26}) + \frac{1}{2}\psi(\Gamma_{37} + \Gamma_{48})}. \quad (20)$$

We compute

$$R^2 = [\cos^2 \vartheta + \frac{1}{2}\mu \sin 2\vartheta + A \sin^2 \vartheta] \times [\cos^2 \psi + \frac{1}{2}\nu \sin 2\psi + B \sin^2 \psi] \quad (21)$$

where

$$\mu = (\Gamma_{15} + \Gamma_{26}), \quad \nu = (\Gamma_{37} + \Gamma_{48}), \quad (22)$$

and

$$A = \frac{1}{2}(\mu^2 + 2), \quad B = \frac{1}{2}(\nu^2 + 2). \quad (23)$$

It is straightforward to show that A and B are both traceless and square to the identity. Moreover, A and B commute with $\Gamma_{12345678}$ and are such that both $\Gamma_{12345678}A$ and $\Gamma_{12345678}B$ are traceless, so the projections of A and B onto the eigenspaces of $\Gamma_{12345678}$ are traceless too. Finally, we observe that

$$AB = \Gamma_{12345678}, \quad (24)$$

so that the product of the eigenvalue a of A with b of B is ± 1 , according to the eigenvalue of $\Gamma_{12345678}$.

With this information we compute that for $ab = 1$

$$R^2 = \begin{cases} 1 & a = 1 \\ e^{\pm 2i(\vartheta + \psi)} \text{ or } e^{\pm 2i(\vartheta - \psi)} & a = -1 \end{cases} \quad (25)$$

where the multiplicities of the $a = \pm 1$ eigenvalues of A are both equal to 8. For $ab = -1$,

$$R^2 = \begin{cases} e^{\pm 2i\vartheta} & a = 1 \\ e^{\pm 2i\psi} & a = -1 \end{cases} \quad (26)$$

where the multiplicities of $a = \pm 1$ eigenvalues of A are again both equal to 8.

At this point we may introduce the constraint (1). We observe that the traceless matrices A and B commute with Γ_{091234} and are such that both $\Gamma_{091234}A$ and $\Gamma_{091234}B$ are traceless too. The same reasoning as before then shows that the projections of A and B onto the subspace satisfying (1) have the same eigenvalues and multiplicities as A and B themselves. This effectively reduces the multiplicities of the $a = +1$ and $a = -1$ eigenvalues of A within each $\Gamma_{12345678}$ eigenspaces from 8 to 4. It follows immediately that for generic angles ϑ and ψ the constraints $\Gamma_{091234}\epsilon = \epsilon$ and $(R^2 - 1)\epsilon = 0$ have just 4 common solutions, corresponding to $a = 1$ in (25). Thus, the generic rotation with R of the form (20) leads to $4/32$, or $1/8$, supersymmetry.

For special values of ϑ and ψ there can be additional solutions of $(R^2 - 1)\epsilon = 0$. For example, when either ϑ or ψ vanishes $(R^2 - 1)$ has four more zero eigenvalues in the subspace satisfying (1). These are the 1/4 supersymmetric solutions discussed above in which two M-5-branes intersect on a 3-plane. When neither ϑ nor ψ are zero but either $\vartheta + \psi$ or $\vartheta - \psi$ is (mod 2π), then four of the eight eigenvalues of $(R^2 - 1)$ corresponding to the $a = -1$ eigenvalue of A in (25) vanish. It is also true, as we saw above, that only four of these eight eigenvalues of $(R^2 - 1)$ correspond to eigenspinors satisfying (1). Unfortunately, it is not obvious from this analysis how many of the latter are zero modes of $(R^2 - 1)$. In fact, just two of them will be because for the given choice of ϑ and ψ the matrix R reduces (up to irrelevant choices of signs of the two angles) to (8), which we earlier showed to lead to 3/16 supersymmetry. Note that if were not for this ambiguity in the current analysis the previous analysis of the 3/16 supersymmetric case would have been unnecessary. The ambiguity does not arise for the special case in which either $\vartheta = \psi = \pm\pi/2$ or $\vartheta = -\psi = \pm\pi/2$ since in that case all eight eigenvalues of $(R^2 - 1)$ corresponding to $a = -1$ in (25) vanish and the issue of which four to choose does not arise. Whichever four we take to satisfy (1) we have a total of 8 common solutions to the M-5-brane constraints and hence 1/4 supersymmetry. This case is of course just the *orthogonal* intersection on a line of two M-5-branes.

5. CONCLUSIONS

I have shown in detail how configurations of non-orthogonally intersecting M-5-branes may preserve 1/4, 3/16, or 1/8 supersymmetry. These results were obtained originally in [1] and [2] by slightly different methods. It is likely that there is a further supersymmetric configuration obtained by a simultaneous rotation in four independent planes, corresponding to rotation within a $Spin(7)$ subgroup of $SO(8)$. If so it should preserve 1/16 supersymmetry. It may also be possible to realize this fraction by a simultaneous rotation in five independent planes, corresponding to an $SU(5)$ rotation in $SO(10)$. It would be desir-

able to have a comprehensive analysis along the above lines to verify this. At any rate, this would appear to be a necessary step for a classification of all supersymmetric M-theory configurations.

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